CS 188: Artificial Intelligence

Lecture 4 and 5: Constraint Satisfaction Problems (CSPs)

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Many slides from Dan Klein

Recap: Search

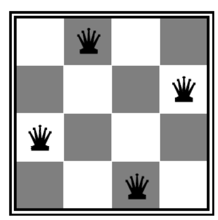
- Search problem:
  - States (configurations of the world)
  - Successor function: a function from states to lists of (state, action, cost) triples; drawn as a graph
  - Start state and goal test
- Search tree:
  - Nodes: represent plans for reaching states
  - Plans have costs (sum of action costs)
- Search Algorithm:
  - Systematically builds a search tree
  - Chooses an ordering of the fringe (unexplored nodes)

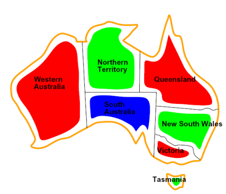
What is Search For?

- Models of the world: single agents, deterministic actions, fully observed state, discrete state space
- Planning: sequences of actions
  - The path to the goal is the important thing
  - Paths have various costs, depths
  - Heuristics to guide, fringe to keep backups
- Identification: assignments to variables
  - The goal itself is important, not the path
  - All paths at the same depth (for some formulations)
  - CSPs are specialized for identification problems

Constraint Satisfaction Problems

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms



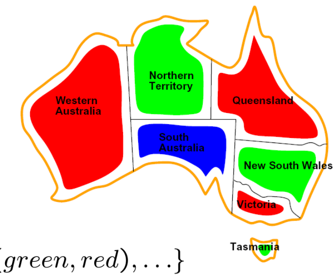


Example CSP: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors

  $WA \neq NT$

  $(WA, NT) \in \{(red, green), (red, blue), (green, red), \ldots\}$



- Solutions are assignments satisfying all constraints, e.g.:

  $\{WA = red, NT = green, Q = red,$
  $NSW = green, V = red, SA = blue, T = green\}$

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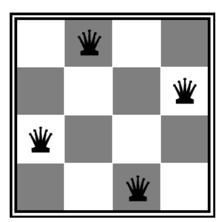
Example CSP: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$

$$\sum_{i,j} X_{ij} = N$$

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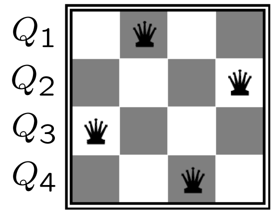
Example CSP: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{1, 2, 3, \dots N\}$
  - Constraints:



Implicit: $\forall i, j$ non-threatening(Q_i, Q_j)

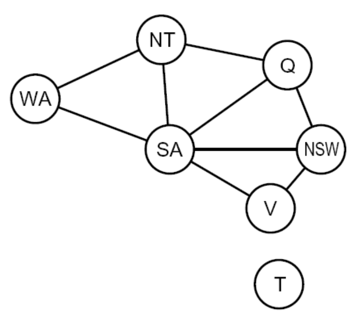
-or-

Explicit: $(Q_1, Q_2) \in \{(1,3), (1,4), \dots\}$

$\dots$

Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Binary constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



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Example CSP: Cryptarithmetic

- Variables (circles):

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$
- Domains:

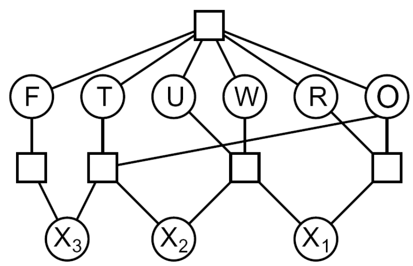
  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Constraints (boxes):

  $\text{alldiff}(F, T, U, W, R, O)$

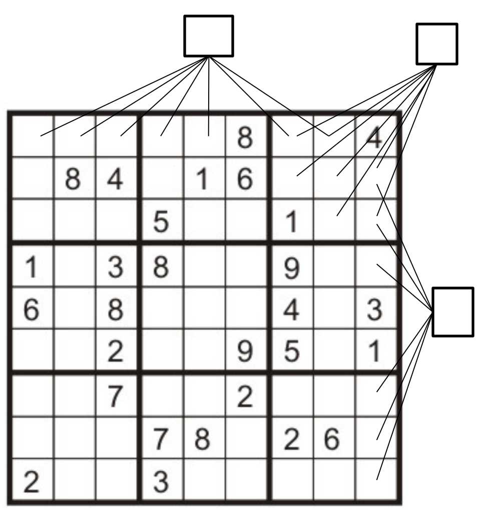
  $O + O = R + 10 \cdot X_1$

  $\cdots$

```
    T W O
  + T W O
  -------
  F O U R
```



Example CSP: Sudoku



- Variables:
  - Each (open) square
- Domains:
  - {1,2,…,9}
- Constraints:

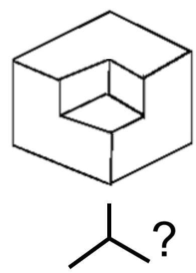
  9-way alldiff for each column

  9-way alldiff for each row

  9-way alldiff for each region

Example CSP: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra
- An early example of a computation posed as a CSP



- Look at all intersections
- Adjacent intersections impose constraints on each other

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Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start-end state of a robot
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)

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Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

    $$SA \neq green$$

  - Binary constraints involve pairs of variables:

    $$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We' ll ignore these until we get to Bayes' nets)

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Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Transportation scheduling
- Factory scheduling
- Floorplanning
- Fault diagnosis
- … lots more!

- Many real-world problems involve real-valued variables…

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Standard Search Formulation

- Standard search formulation of CSPs (incremental)
- Let's start with the straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints
- Simplest CSP ever: two bits, constrained to be equal

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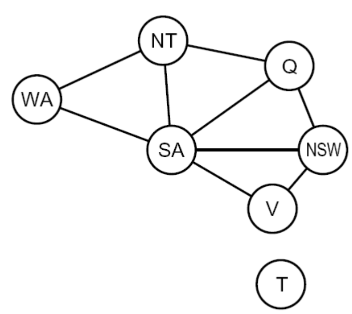
Search Methods

- What does BFS do?



- What does DFS do?
  - [demo]
- What’s the obvious problem here?
- What’s the slightly-less-obvious problem?

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Backtracking Search

- Idea 1: Only consider a single variable at each point
  - Variable assignments are commutative, so fix ordering
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there?
- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok
  - "Incremental goal test"
- Depth-first search for CSPs with these two improvements is called *backtracking search* (useless name, really)
  - [DEMO]
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

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Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
   return RECURSIVE-BACKTRACKING({ }, csp)
function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
   if assignment is complete then return assignment
   var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
   for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
       if value is consistent with assignment given CONSTRAINTS[csp] then
           add {var = value} to assignment
           result ← RECURSIVE-BACKTRACKING(assignment, csp)
           if result ≠ failure then return result
           remove {var = value} from assignment
   return failure
```

- Backtracking = DFS + var-ordering + fail-on-violation
- What are the choice points?

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Improving Backtracking

- General-purpose ideas give huge gains in speed
- Ordering:
  - Which variable should be assigned next?
  - In what order should its values be tried?
- Filtering: Can we detect inevitable failure early?
- Structure: Can we exploit the problem structure?

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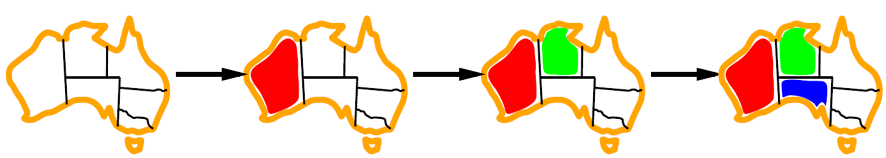
Minimum Remaining Values

- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values



- Why min rather than max?
- Also called “most constrained variable”
- Also called “fail-fast” ordering

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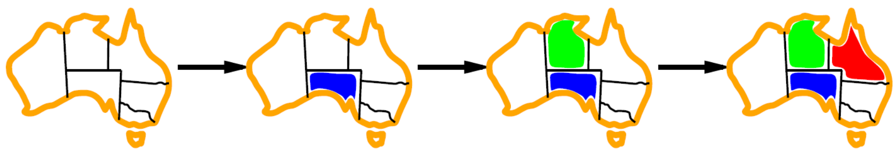
Degree Heuristic

- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable participating in the most constraints on remaining variables



- Why most rather than fewest constraints?

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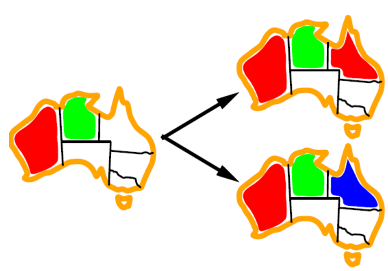
Least Constraining Value

- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!



- Why least rather than most?

- Combining these heuristics makes 1000 queens feasible

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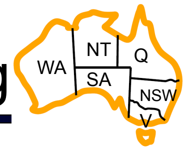
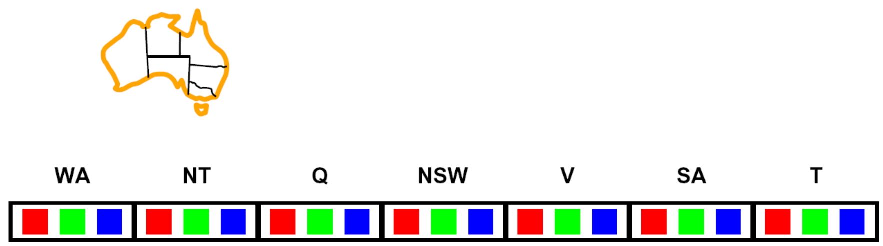
Filtering: Forward Checking



- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints)
- Idea: Terminate when any variable has no legal values



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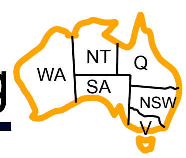
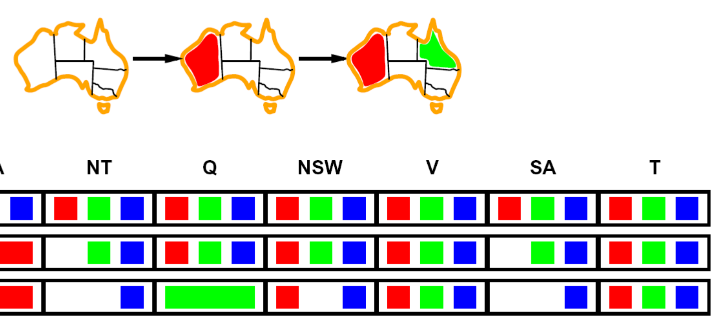
[demo: forward checking animation]

Filtering: Forward Checking



- Forward checking propagates information from assigned to adjacent unassigned variables, but doesn't detect more distant failures:



- NT and SA cannot both be blue!
- Why didn’t we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

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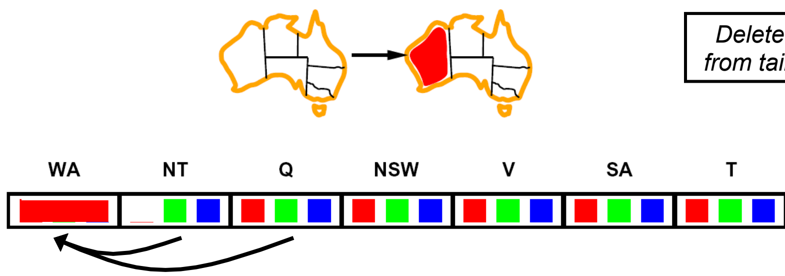
Consistency of An Arc

- An arc X → Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating a constraint

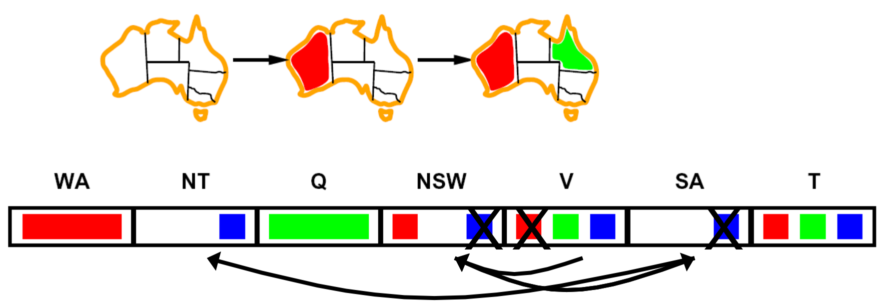
Delete from tail!



- What happens?
- Forward checking = Enforcing consistency of each arc pointing to the new assignment

Arc Consistency of a CSP

- Simplest form of propagation makes each arc *consistent*
  - X → Y is consistent iff for *every* value x there is *some* allowed y



- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What's the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

Establishing Arc Consistency

```
function AC-3( csp) returns the CSP, possibly with reduced domains
   inputs: csp, a binary CSP with variables {X1, X2, ..., Xn}
   local variables: queue, a queue of arcs, initially all the arcs in csp

   while queue is not empty do
      (Xi, Xj) ← REMOVE-FIRST(queue)
      if REMOVE-INCONSISTENT-VALUES(Xi, Xj) then
         for each Xk in NEIGHBORS[Xi] do
            add (Xk, Xi) to queue
─────────────────────────────────────────────
function REMOVE-INCONSISTENT-VALUES( Xi, Xj) returns true iff succeeds
   removed ← false
   for each x in DOMAIN[Xi] do
      if no value y in DOMAIN[Xj] allows (x,y) to satisfy the constraint Xi ↔ Xj
         then delete x from DOMAIN[Xi];  removed ← true
   return removed
```

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

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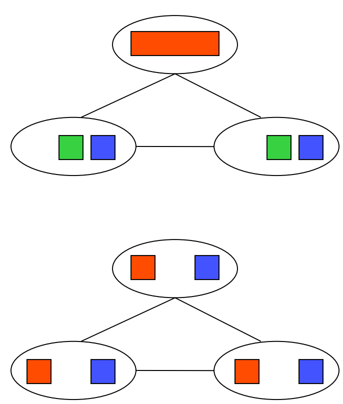
[demo: arc consistency animation]

Limitations of Arc Consistency

- After running arc consistency:
  - Can have one solution left
  - Can have multiple solutions left
  - Can have no solutions left (and not know it)



What went wrong here?

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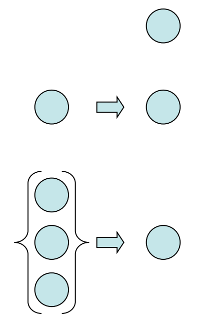
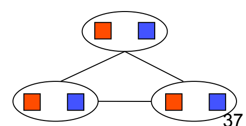
K-Consistency

- Increasing degrees of consistency
  - 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints
  - 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other
  - K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the k^{th} node.



- Higher k more expensive to compute
- (You need to know the k=2 algorithm)



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Strong K-Consistency*

- Strong k-consistency: also k-1, k-2, … 1 consistent
- Claim: strong n-consistency means we can solve without backtracking!
- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …
- Lots of middle ground between arc consistency and n-consistency! (e.g. path consistency)

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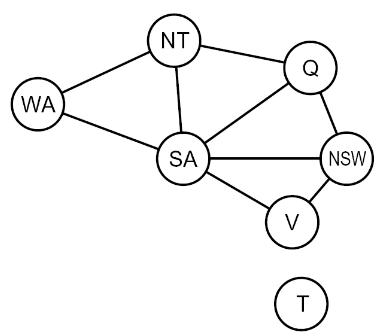
Backtracking with MRV, Degree, LCV, Filtering

```
function RecursiveBacktracking(pa, fd, vars, constraints)
    if  IsComplete(pa)  then return pa
    next_var <-- select_MRV_Degree(pa, fd, vars, constraints)
    for each value in fd[next_var] do
        new_fd[value] <-- constraint_prop(pa, fd, vars, constraints)
    for each value in fd[next_var] in order of LCV do
        if any of the domains in new_fd[value] is empty
            continue;
        else  // all domains in new_fd[value] have at least one value remaining
            add {var=value} to pa
            result <-- recursive_backtracking(pa, new_fd[value], vars, constraints)
            if (result not equal to failure) then return result
    //if we get here none of the expansions led to a solution
    return failure
```

- select_MRV_degree: selects an unassigned variable based on MRV an degree heuristic
- constraint_prop:performs constraint propagation, this could be through forward propagation or through arc consistency
- pa: partial assignment
- fd: filtered domains

Problem Structure

- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
  - Worst-case solution cost is O((n/c)(d^c)), linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - (4)(2^{20}) = 0.4 seconds at 10 million nodes/sec



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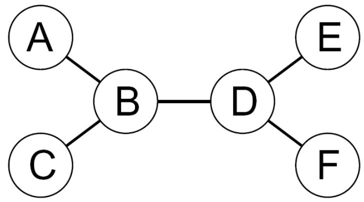
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to probabilistic reasoning (later): an important example of the relation between syntactic restrictions and the complexity of reasoning.

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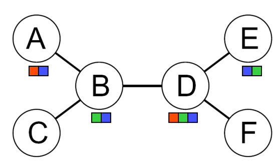
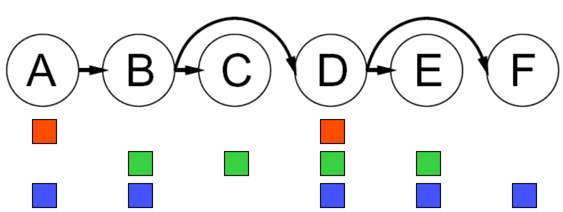
Tree-Structured CSPs

- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering





- For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
- For i = 1 : n, assign X_i consistently with Parent(X_i)
- Runtime: $O(n\ d^2)$ (why?)

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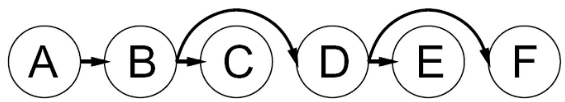
Tree-Structured CSPs

- Why does this work?
- Claim: After processing the right k nodes, given any satisfying assignment to the rest, the right k can be assigned (left to right) without backtracking.
- Proof: Induction on position



- Why doesn't this algorithm work with loops?
- Note: we'll see this basic idea again with Bayes' nets

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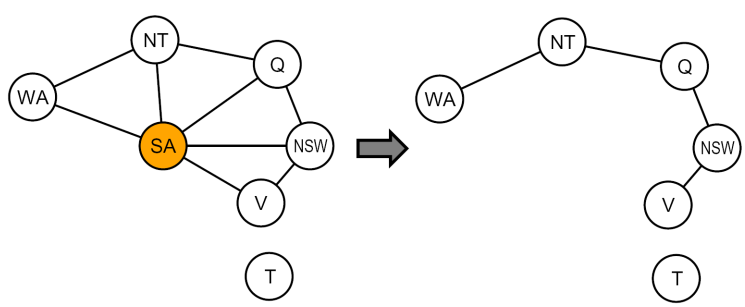
Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

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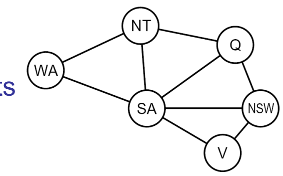
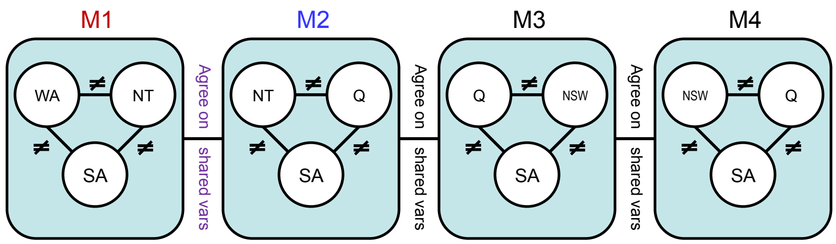
Tree Decompositions*

- Create a tree-structured graph of overlapping subproblems, each is a mega-variable
- Solve each subproblem to enforce local constraints
- Solve the CSP over subproblem mega-variables using our efficient tree-structured CSP algorithm





{(WA=r,SA=g,NT=b), (WA=b,SA=r,NT=g), …}

{(NT=r,SA=g,Q=b), (NT=b,SA=g,Q=r), …}

Agree: (M1,M2) ∈ {((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}

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CSPs: our status

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search with
  - Branching on only one variable per layer in search tree
  - Incremental constraint checks ("Fail fast")
- Heuristics at our points of choice to improve running time:
  - Ordering variables: Minimum Remaining Values and Degree Heuristic
  - Ordering of values: Least Constraining Value
  - Filtering: forward checking, arc consistency → enable computation of these heuristics
- Structure: Disconnected and tree-structured CSPs are efficient
- Iterative improvement

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Iterative Algorithms for CSPs

- Local search methods typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - Start with some assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

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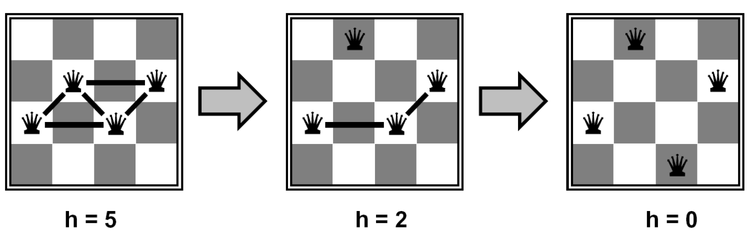
Example: 4-Queens



- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks, i.e., no two queens on same row, same column or same diagonal
- Evaluation: c(n) = number of attacks

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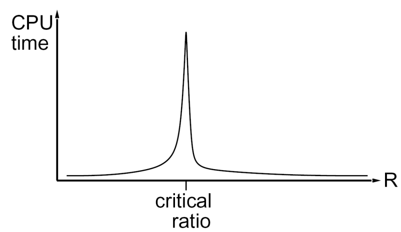
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



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Hill Climbing

- Simple, general idea:
  - Start wherever
  - Always choose the best neighbor
  - If no neighbors have better scores than current, quit
- Why can this be a terrible idea?
  - Complete?
  - Optimal?
- What's good about it?

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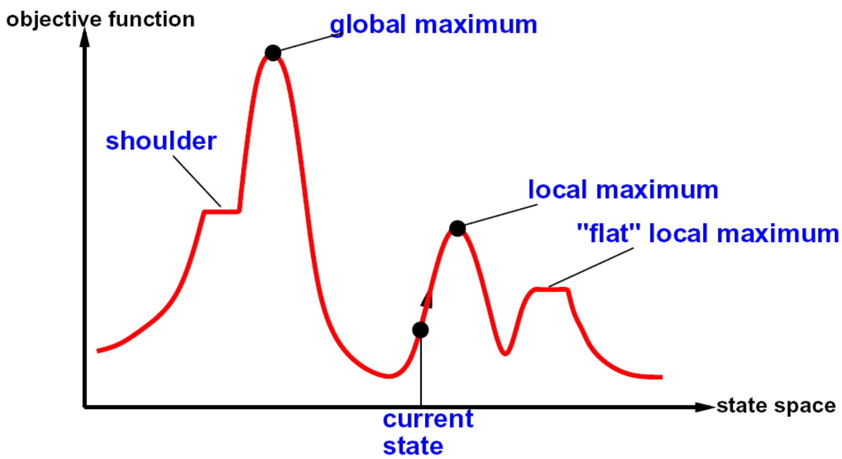
Hill Climbing Diagram



- Random restarts?
- Random sideways steps?

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Simulated Annealing*

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

function SIMULATED-ANNEALING(*problem*, *schedule*) **returns** a solution state
  inputs: *problem*, a problem
    schedule, a mapping from time to "temperature"
  local variables: *current*, a node
    next, a node
    T, a "temperature" controlling prob. of downward steps

  current ← MAKE-NODE(INITIAL-STATE[*problem*])
  for $t \leftarrow 1$ **to** ∞ **do**
    $T \leftarrow$ *schedule*[t]
    if $T = 0$ **then return** *current*
    next ← a randomly selected successor of *current*
    $\Delta E \leftarrow$ VALUE[*next*] – VALUE[*current*]
    if $\Delta E > 0$ **then** *current* ← *next*
    else *current* ← *next* only with probability $e^{\Delta E/T}$

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Simulated Annealing*

- Theoretical guarantee:
  - Stationary distribution: $p(x) \propto e^{\frac{E(x)}{kT}}$
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee?
- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape, the less likely you are to ever make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways

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Recap CSPs

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search (why?, tree or graph search?) with
  - Branching on only one variable per layer in search tree
  - Incremental constraint checks ("Fail fast")
- Heuristics at our points of choice to improve running time:
  - Ordering variables: Minimum Remaining Values and Degree Heuristic
  - Ordering of values: Least Constraining Value
  - Filtering: forward checking, arc consistency → computation of heuristics
- Structure: Disconnected and tree-structured CSPs are efficient
  - Non-tree-structured CSP can become tree-structured after some variables have been assigned values
- Iterative improvement: min-conflicts is usually effective in practice

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